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Strong Parity Weighted Totally Antimagic Total (SPAT) Graphs

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Abstract

An $1-1$ correspondence mapping $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, p+q\}$ is a total labeling of a finite undirected graph G without loops and multiple edges, where $p = |V(G)|$ and $q = |E(G)|$. A Perfectly Antimagic Total (PAT) labeling is a Totally Antimagic Total (TAT) labeling in which each vertex weight is also pairwise distinct from each of its edge weights. In this paper, we introduce a new parameter called strong parity weighted labeling. A TAT labeling is a strong parity weighted TAT (SPAT) labeling if all the vertex (edge) weights are distinct even (odd) integers. A graph that admits such labeling is called a strong parity weighted TAT (SPAT) graph. Our findings established that several well-known families of graphs, including cycles, paths, stars, complete graphs, bi-stars, and ladders, admit SPAT labeling. We first illustrate the SPAT labeling for these families utilizing existing methodologies in labeling theory. Furthermore, we develop novel techniques that extend the analysis to other graph families, determining their potential to admit SPAT labeling.

Keywords: antimagic labeling; totally antimagic total graphs; perfectly antimagic total graphs.

1 Introduction

Graph labeling is a well-established area of graph theory, where labels are assigned to elements of graphs according to specific rules. In this paper, we focus on undirected, finite graphs without loops or multiple edges, denoted as $G = (V(G), E(G))$, where $p = |V(G)|$ and $q = |E(G)|$ represents the order and size of G, respectively. The neighbours of a vertex $u \in V(G)$ are denoted as $N(u)$. Vertex labeling is used when the domain is only $V(G)$. Labeling is referred to as edge labeling if the domain is a mere $E(G)$. When the labeling's domain is $V(G) \cup E(G)$, it is referred to as total labeling. A total labeling of graph G is an one to one correspondence function $\lambda : V(G) \cup E(G) \to \{1, 2, 3, \dots, p + q\}$. Such a labeling λ is vertex antimagic total (edge antimagic total) if all its vertex weights,

$$wt_\lambda(u) = \lambda(u) + \sum_{v \in N(u)} \lambda(uv), \qquad u \in V(G),$$

$\big($all its edge weights $wt_\lambda(uv) = \lambda(u) + \lambda(uv) + \lambda(v)$, $uv \in E(G)\big)$ are pairwise distinct. If total labeling is simultaneously vertex antimagic total and edge antimagic total, then it is called TAT labeling. We monitor the updated version of Gallian's dynamic survey [5] for new developments in graph labeling.

The idea of antimagic (edge) labeling, which is fundamentally distinct from TAT labeling, was first presented by Hartsfield and Ringal in [6]. Exoo et al. [4] suggested the idea of totally magic labeling and also refer to [9]. Researchers studying labeling theory have realized that TAT labeling is an antipodal variation of totally magic labeling. Miller et al. [10] establishes that all graphs admit antimagic total, super antimagic total, and repus antimagic total labelings, and further demonstrates the existence of (c, d)-antimagic total labelings where vertex weights form an arithmetic progression. Bača et al. [1, 2] and associates developed the idea of TAT labeling of graphs in 2015, proving both its existence and nonexistence for particular families of graphs. Ivančo [8] introduces a broad class of totally antimagic total graphs by establishing total labelings where both vertex and edge weights are pairwise distinct. Hasni et al. [7] studied edge irregular k-labeling for disjoint unions of cycles and generalized prisms, contributing to graph labeling techniques. Yoong et al. [12] explored edge irregular reflexive labeling for certain plane graphs, offering useful insights for antimagic total labeling.

The PAT labeling [11], a further refinement of TAT labeling, addresses whether vertex and edge weights can be distinct from each other and themselves. Balasundar et al. [3] investigate PAT labeling and its variant, Strongly Vertex Perfectly Antimagic Total labeling (SVPAT), proving that not all trees admit SVPAT and that it is exclusively achievable in path graphs. To address this gap, we introduce a novel concept: Strong Parity Weighted Totally Antimagic Total (SPAT) labeling. In this labeling, vertex weights are distinct even numbers, and edge weights are different odd numbers. This builds parity-based labeling, which enhances complexity as well as uniqueness in the labeling system.

The introduction of parity-weighted conditions within the TAT framework is a unique aspect of our research. We enhance the labeling in conventional TAT to provide greater flexibility and control over label assignments, allowing the investigation of additional classes of graphs that permit such labeling. This method is more rigorous and systematic for graph labeling. It would facilitate a wider array of applications, including network architecture and cryptography, where varying even and odd limitations are significant. We also present the robust parity-weighted TAT for many prominent graph families: cycle, path, star, bi-star, complete, and ladder graphs. These examples demonstrate that our labeling approach is adaptable and outperforms earlier methods based on distinctness while also meeting parity criteria.

The primary contribution of this paper is the introduction of SPAT labeling and the demonstration of its applicability to several important graph classes. We present a new approach that advances the field of graph labeling by providing a method that is both more structured and adaptable than existing TAT labelings.

2 Main Results

Definition 2.1. *A TAT graph is said to be a strong parity weighted TAT graph if the set of vertex weights consists of the pairwise distinct even integer and the set of edge weights consists of the set of pairwise distinct odd integers.*

Theorem 2.1. *Every cycle C_n $(n \geq 3)$ is strong parity weighted TAT graph.*

Proof. Let C_n be a cycle graph that consists of n vertices, where $n \geq 3$ and each vertex has degree 2. Define the vertex and edge set as follows,

$$\begin{aligned} V(C_n) &= \{v_1, v_2, v_3, \dots, v_n\}, \\ E(C_n) &= \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}. \end{aligned}$$

Define a map $\lambda : V(C_n) \cup E(C_n) \to \{1, 2, 3, \dots, 2n\}$ in the following way,

$$\begin{aligned} \lambda(v_i) &= 2i, && 1 \leq i \leq n, \\ \lambda(v_i v_{i+1}) &= 2n - (2i - 1), && 1 \leq i \leq n-1, \\ &= 2(n - i) + 1, && 1 \leq i \leq n-1, \\ \lambda(v_1 v_n) &= 1. \end{aligned}$$

For $i = 1$,

$$\begin{aligned} wt_\lambda(v_1) &= \lambda(v_1) + \lambda(v_1 v_2) + \lambda(v_1 v_n) \\ &= 2 + 2n - 1 + 1 \\ &= 2n + 2. \end{aligned}$$

For $i = 2, 3, \dots, n-1$,

$$\begin{aligned} wt_\lambda(v_i) &= \lambda(v_i) + \sum_{v_j \in N(v_i)} \lambda(v_i v_j) \\ &= \lambda(v_i) + \lambda(v_i v_{i+1}) + \lambda(v_{i-1} v_i) \\ &= 4(n + 1) - 2i. \end{aligned}$$

For $i = n$,

$$\begin{aligned} wt_\lambda(v_n) &= \lambda(v_n) + \lambda(v_{n-1} v_n) + \lambda(v_n v_1) \\ &= 2n + 2n - (2(n - 1) - 1) + 1 \\ &= 2n + 4. \end{aligned}$$

For $1 \leq i \leq n-1$,

$$\begin{aligned} wt_\lambda(v_i v_{i+1}) &= \lambda(v_i) + \lambda(v_i v_{i+1}) + \lambda(v_{i+1}) \\ &= 2i + 2(n-i) + 1 + 2(i+1) \\ &= 2n + 2i + 3, \\ wt_\lambda(v_n v_1) &= \lambda(v_n) + \lambda(v_n v_1) + \lambda(v_1) \\ &= 2n + 1 + 2 \\ &= 2n + 3. \end{aligned}$$

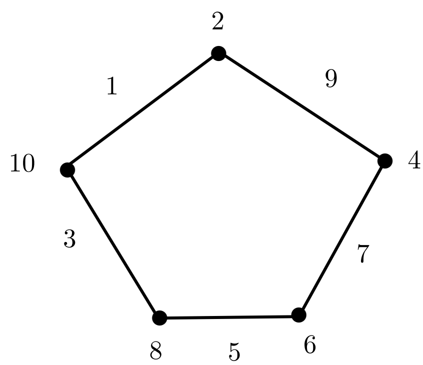

Figure 1: Cycle C_5.

Using this labeling structure, as illustrated in Figure 1, we obtain C_n for $n \geq 3$ in the TAT graph because all of the vertex weights and edge weights are pairwise distinct. The set of pairwise distinct even integers compensates the set $\{wt_\lambda(v_i) : i = 1, 2, \ldots, n\}$ as well. Additionally, the set of pairwise different odd integers is represented by the set,

$$\{wt_\lambda(v_i v_{i+1}) : i = 1 \leq i \leq n-1\} \cup wt_\lambda(v_n v_1).$$

As a result, the strong parity weighted TAT graph is C_n for $n \geq 3$. □

Theorem 2.2. *Every path P_n, $n > 1$, $n \not\equiv 0(\mod 3)$ is not strong parity weighted TAT graph.*

Proof. Let $P_n = v_1 e_1 v_2 e_2 \ldots v_{n-1} e_{n-1} v_n$ be a path of length n with n vertices and $(n-1)$ edges.

Case (i) For $n > 1$ and $n \equiv 2(\mod 3)$, assume P_n is a strong parity weighted TAT graph. Then, there is a TAT labeling where the edge weights are odd and the vertex weights are even. The pendant vertices (v_1, v_n) can only have labels of either both odd or both even, since each vertex weight is even. If,

$$[\lambda(v_1), \lambda(e_1), \lambda(v_2)] = [\text{odd, odd, odd}], \quad [\lambda(v_{n-1}), \lambda(e_{n-1}), \lambda(v_n)] = [\text{odd, odd, odd}],$$

and $n+1 \equiv 0(\mod 3)$, then $n+1$ odd labels are needed to assign parity weighted TAT.

However, we can only have n odd labels from the domain set,

$$\{1, 2, \ldots, p+q\} = \{1, 2, \ldots, 2n-1\},$$

which leads to a contradiction. Therefore, for $n \equiv 2(\mod 3)$ and n more than 1, P_n is not a strong parity weighted TAT graph. Beginning with,

$$[\lambda(v_1), \lambda(e_1), \lambda(v_2)] = [\text{even, even, odd}],$$

then, $\lambda(e_{n-1})$ must be even but $\lambda(v_n)$ can not be label with either odd (or) even. If $\lambda(v_n)$ = odd, then the edge weight of e_{n-1} is odd but the vertex weight of v_n is odd, which gets a contradiction. If $\lambda(v_n)$ = even, then the vertex weight is even but the edge weight e_{n-1} is even, we get a contradiction. Hence, the path graph P_n is not strong parity weighted TAT graph for $n \equiv 2(\mod 3)$.

Case (ii) If $n > 1$, let $n \equiv 1(\mod 3)$. Then, we suppose that P_n is a strong parity weighted TAT graph in the opposite direction. Then, each vertex weight is even under a TAT labeling λ, which is present. The labels $(\lambda(v_1), \lambda(e_1))$ can only receive (odd, odd) or (even, even) labels because each vertex weight is even. If $(\lambda(v_1), \lambda(e_1))$ =(odd, odd), then $\lambda(v_n)$ must meet either of the two conditions ($wt_\lambda(v_n)$ = even or $wt_\lambda(e_{n-1})$ = odd). However, $\lambda(v_n)$ cannot be categorized as either even or odd.

Therefore, there is a contradiction because the TAT labeling λ is not strongly parity weighted. The required number of odd labels is smaller than the actual number of odd labels if $(\lambda(v_1), \lambda(e_1))$ =(even, even). If not, there are more necessary even labels than there are genuine even labels. We now encounter a contradiction.

Therefore, for $n \equiv 1(\mod 3)$ and n more than 1, the path graph P_n is not strongly parity weighted.

□

Theorem 2.3. *For $n \equiv 0(\mod 3)$, the path P_n is strong parity weighted TAT graph.*

Proof. Let us start by assigning the labels for $\lambda(v_1)$, $\lambda(e_1)$ and $\lambda(v_2)$ as odd, odd, and odd, respectively. Now, we define a total labeling $\lambda : V(P_n) \cup E(P_n) \to \{1, 2, \ldots, 2n-1\}$ as follows,

$$\begin{aligned}
\lambda(v_1) &= 1,\\
\lambda(v_i) &= \lambda(e_{i-1}) + \left\lceil \frac{2n-1}{3} \right\rceil (\mod 2n-1), \quad \text{for } i = 2, 3, \ldots, n,\\
\lambda(e_1) &= 1 + \left\lceil \frac{2n-1}{3} \right\rceil (\mod 2n-1),\\
\lambda(e_i) &= \lambda(v_i) + \left\lceil \frac{2n-1}{3} \right\rceil (\mod 2n-1), \quad \text{for } i = 2, 3, \ldots, n.
\end{aligned}$$

Now,

$$\begin{aligned}
wt_\lambda(v_1) &= 2 + \left\lceil \frac{2n-1}{3} \right\rceil,\\
wt_\lambda(e_1) &= 3 + 3\left\lceil \frac{2n-1}{3} \right\rceil,\\
wt_\lambda(v_i) &= wt_\lambda(e_{i-1}) + 1, \quad \text{for } i = 2, 3, \ldots, n-1,\\
wt_\lambda(e_i) &= wt_\lambda(v_i) + 1, \quad \text{for } i = 2, 3, \ldots, n-1,\\
wt_\lambda(v_n) &= 2\left\lceil \frac{2n-1}{3} \right\rceil + \lambda(e_{n-1}).
\end{aligned}$$

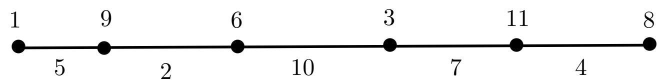

Figure 2: Path P_6.

Following the above labeling, the path graph, as demonstrated in Figure 2. Now, we can easily check that,

$$wt_\lambda(v_1) < wt_\lambda(v_n) < wt_\lambda(e_1) < wt_\lambda(v_2) < wt_\lambda(e_2) < \ldots < wt_\lambda(v_{n-1}) < wt_\lambda(e_{n-1}).$$

In the case of $n \equiv 0(\mod 3)$, $\left\lceil \frac{2n-1}{3} \right\rceil$ is always even. Now, it is simple to verify that all edge weights and all vertex weights are pairwise distinct odd and even integers, respectively. As a result, P_n is labeled with a strong parity weighted TAT for $n \equiv 0(\mod 3)$.

Thus, when $n \equiv 0(\mod 3)$, P_n is a strong parity weighted TAT graph. □

Theorem 2.4. *The star $K_{1,n}$ is a strong parity weighted TAT graph if and only if $n \equiv 2(\mod 4)$.*

Proof. Let $K_{1,n}$ be a star graph with $n+1$ vertices and n edges, where the central vertex has a degree of n, while all other vertices have a degree of 1. The vertex and edge set are defined as follows,

$$V(K_{1,n}) = \{v_i : 1 \le i \le n+1\},$$
$$E(K_{1,n}) = \{(v_1, v_i) = e_{i-1} : 2 \le i \le n+1\},$$

and v_1 is called a central vertex and all other vertices are called pendent vertices.

Now, $p = |V(K_{1,n})| = n+1$ and $q = |E(K_{1,n})| = n$. Hence, $p+q = 2n+1$.

For the star $|V(K_{1,n})| + |E(K_{1,n})| = 2n+1$ and the set $\{1, 2, \ldots, 2n+1\}$ contains n even integers and $n+1$ odd integers.

As under any strong parity weighted TAT labeling of $K_{1,n}$ the vertex weights must be even we get that the label of every edge must have the same parity as the label of the incident vertex of degree 1, (i.e., they are both even or both odd). This implies that n must be even and also the central vertex must have an odd label.

Moreover, as the weight of the central vertex must be also even and this weight is the sum of $\frac{n}{2}$ even edge labels, $\frac{n}{2}$ odd edge labels and label of the central vertex (which must be an odd integer) we get that $\frac{n}{2}$ must be odd. Thus, when $n \not\equiv 2(\mod 4)$ the star $K_{1,n}$ is not strong parity weighted TAT.

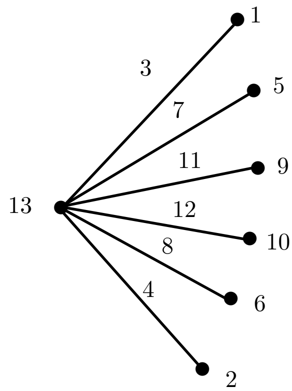

Figure 3: Star $K_{1,6}$.

Conversely, assume $n \equiv 2 (\mod 4)$.
Define a bijective map $\lambda : V(K_{1,n}) \cup E(K_{1,n}) \to \{1, 2, 3, \dots, 2n+1\}$ in the following way,

$$\begin{aligned}
\lambda(v_1) &= 2n+1, \\
\lambda(v_2) &= 1, \\
\lambda(v_s) &= 4s-7, & 3 \le s \le \frac{n}{2}+1, \\
\lambda(v_j) &= 2(2n-2j+3), & \frac{n}{2}+2 \le j \le n+1, \\
\lambda(e_i) &= 4i-1, & 1 \le i \le \frac{n}{2}, \\
\lambda(e_i) &= 4(n+1-i), & \frac{n}{2}+1 \le i \le n.
\end{aligned}$$

Next, the vertex weights are given by,

$$\begin{aligned}
wt_\lambda(v_1) &= \frac{1}{2}(2n^2+7n+2), && \text{which is an even.} \\
wt_\lambda(v_2) &= 4, && \text{which is even.} \\
wt_\lambda(v_i) &= 4i-7+4(i-1)-1, \\
wt_\lambda(v_i) &= 4(2i-3), && \text{where } \; 3 \le i \le \frac{n}{2}+1.
\end{aligned}$$

Clearly $wt_\lambda(v_i)$ is even, for $3 \le i \le \frac{n}{2}+1$.

For $\frac{n}{2}+2 \le i \le n+1$, we have,

$$wt_\lambda(v_i) = 2(4n-4i+7), \qquad \frac{n}{2}+2 \le i \le n+1.$$

Clearly, $wt_\lambda(v_i)$ is even, for $\frac{n}{2}+2 \le i \le n+1$.
Now, we find the edge weights as follows,

$$\begin{aligned}
wt_\lambda(e_1) &= 2n+5, && \text{which is an odd.} \\
wt_\lambda(e_i) &= 4(i+1)-7+(4i-1)+(2n+1), \\
wt_\lambda(e_i) &= 2(4i+n)-3, && 2 \le i \le \frac{n}{2}.
\end{aligned}$$

Clearly, $wt_\lambda(e_i)$ is an odd for $2 \leq i \leq \frac{n}{2}$.

For $\frac{n}{2} + 1 \leq i \leq n$, we have,

$$wt_\lambda(e_i) = 2(2n - 2(i+1) + 3) + 4(n + 1 - i) + 2n + 1,$$
$$wt_\lambda(e_i) = 2(5n - 4i) + 7, \frac{n}{2} + 1 \leq i \leq n.$$

Clearly $wt_\lambda(e_i)$ is an odd for $\frac{n}{2} + 1 \leq i \leq n$.

Using this labeling structure, as shown in Figure 3. Since all the vertex weights are even and all the edge weights are odd, the star graph $K_{1,n}$ is parity weighted TAT graph for $n \equiv 2(\mod 4)$ and $n > 1$. □

Theorem 2.5. *For any positive integer k, the complete graph K_{2k} is not a strong parity weighted TAT graph.*

Proof. Suppose that the complete graph K_{2k} is strong parity weighted TAT for some $k \in Z^+$. Then, there exists a TAT labeling $\lambda : V(K_{2k}) \cup E(K_{2k}) \to \{1, 2, 3, \ldots, k(2k+1)\}$ such that each vertex weight is even and each weight is odd.

Let, $V(K_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(K_n) = \{v_iv_j : i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n, i \neq j\}$. Fix v_1 so that $\lambda(v_1)$ is an odd vertex. There must be a $2l + 1$ incident edges of v_1 that receive the distinct odd labels to comply with the vertex weight requirement. Finally, the distinct even labels must be applied to the remaining $n - 2l - 2$ incident edges. Since the TAT labeling λ is strongly parity weighted, both the vertex weight and edge weight properties must be satisfied. Now, the incident edges of the vertices with odd labels must receive $n - 2l - 2$ even labels and $2l + 2$ odd labels, respectively. However, the vertices which are having even labels and their incident edges must receive the $n - 2l - 3$ number of odd labels and $2l + 3$ number of even labels,

$$\begin{aligned} n - 2l - 3 &= 2k - 2l - 3, \\ &= 2(k - l) - 3, \quad \text{which is always an odd.} \end{aligned}$$

We have an odd number of odd labels and an odd number of even labels at the vertices of an even label and its incident edges. Therefore, not all of the vertices with even labels satisfy the vertex weight property. This runs counter to the TAT labeling's significant parity weight. Therefore, for any $k \in Z^+$, K_{2k} is not a strong parity weighted TAT graph. □

Theorem 2.6. *The bi-star graph $B_{n,n}$ is not a strong parity weighted TAT graph for all $n \geq 1$.*

Proof. Let, $V(B_{n,n}) = \{u, u_1, u_2, \ldots, u_n, v, v_1, v_2, \ldots, v_n\}$ and
$E(B_{n,n}) = \{e_1 : e_1 = uv\} \cup \{e_{i+1} : e_{i+1} = uu_i, 1 \leq i \leq n\} \cup \{e_{i+n+1} : e_{i+n+1} = vv_i, 1 \leq i \leq n\}$, be the vertex and edge set of bi-star graph, respectively. The order and size of a bi-star graph is $2(n+1)$ and $2n+1$, respectively and $S = \{1, 2, 3, \ldots, p+q\} = \{1, 2, 3, \ldots, 4n+3\}$.

Assume that $B_{n,n}$ is a parity weighted TAT graph for all $n \geq 1$. Then, there is a TAT labeling $\lambda : V(B_{n,n}) \bigcup E(B_{n,n}) \to S$ such that each vertex weight is even and each edge weight is odd. Without loss of generality, one can assume that $\lambda(u) + \lambda(v)$= odd (or) even. If $\lambda(u) + \lambda(v) =$ odd, then $\lambda(u)$ =odd, $\lambda(v)$ =even (or) $\lambda(u) =$ even, $\lambda(v)$ =odd, only possible to discuss.

In this case, without loss of generality, we may assume $\lambda(u) =$ odd, $\lambda(v) =$ even only. If $\lambda(u) + \lambda(v) =$ even, then either $\lambda(u)$ and $\lambda(v)$ are odd (or) $\lambda(u)$ and $\lambda(v)$ are even only possible.

Case (i): When n is odd. We assume that $n = 2k + 1$ for some $k \in Z^+$.

When $\lambda(u)$ is odd and $\lambda(v)$ is even, $\lambda(e_1)$ must also be odd. Given that $\lambda(u)$ is odd and $\lambda(e_1)$ is even, it takes an odd number of odd labels to label the edges $\lambda(e_i)$ for the formula $2 \leq i \leq 2k+2$. Let us take $(2s+1)$ number of $\lambda(e_i)$'s are odd, the remaining $2(k-s)$ number of $\lambda(e_i)$'s are even. We note that $1 \leq 2s+1 \leq 2k+1$ and $1 \leq 2(k-s) \leq 2k+1$. Hence, its $2(k-s)$ pendant vertices receives even labels. Since $\lambda(u)$ is odd and $(2s+1)$ number of $\lambda(e_i)$ are odd, then its pendant vertex labels $\lambda(u_i)$ must be odd.

Since $\lambda(v)$ is even and $\lambda(e_1)$ is also even, there are only $2t$ incident edges that receive odd labels for some $t \in Z^+$. We note that $1 \leq 2t \leq 2k+1$. Now, the remaining labels of $2(k-t)+1$ edges incident with v are even. Hence, the 2t number of its pendant vertices couldn't receive either odd or even. Moreover, $2(k-t)+1$ vertices also don't get either odd or even. Hence, we get a contradiction.

If both $\lambda(u)$ and $\lambda(v)$ are odd, then $\lambda(e_1)$ must be odd. Since $\lambda(u), \lambda(v), \lambda(e_1)$ are odd, there must be a $2l$ number of edge labels $\lambda(e_i)$ which are incident with u are odd for some $k \in Z^+$ and $2 \leq i \leq n+1$. The remaining $2(k-l)+1$ incident edges $f(e_i)$ are even for $2 \leq i \leq n+1$. Now, the only possibility of $2l$ pendant vertex labels$f(u_i)$ are odd. But, the remaining $2(k-l)+1$ pendant vertex labels $\lambda(u_i)$ are even. Since $\lambda(v)$ and $\lambda(e_1)$ are odd, there are $2s$ edge labels are odd and its $2s$ pendant vertex labels are odd.

Similarly, we have $2(k-s)+1$ edge labels are even and its $2(k-s)+1$ pendant vertex labels are even. Hence, the total number of required odd labels to assign $= 4(l+s)+2$ and the total number of required even labels to assign $0 = 4(2k-l-s)+4$. But from the range set S, the total number of actual odd labels available $= 4(k+1)$ and the total number of actual even labels available $= 4k+3$. Since $4k+3 \neq 4(2k-l-s)+4$, the number of actual even labels and the number of available even labels are different, we get a contradiction to the existence of λ. If $\lambda(u)$ and $\lambda(v)$ are even, then e_1 must receive an odd label. Since $\lambda(u)$ is even and $\lambda(e_i)$ is odd, we have $(2w+1)$ incident edges of u other than e_1 receive odd labels and their $(2w+1)$ pendant vertices couldn't get either by even or odd labels, get a contradiction.

Case (ii): Whenever n is even. For some $m \in Z^+$, we consider $n = 2m$.

Similar justifications are mentioned in **Case (i)**. When $\lambda(u)$ is odd and $\lambda(u)$ is even, $\lambda(e_1)$ must also be odd. The incident edges of u must receive $(2r+1)$ odd labels because $\lambda(u)$ is odd, and its corresponding $(2r+1)$ pendant vertices must likewise receive odd labels.

Now, the remaining incident $2(m-r)+1$ edges of u receive even labels and its $2(m-r)+1$ pendant vertices must have even labels. Since $\lambda(v)$ is even and $\lambda(e_1)$ is even, $2x$ incident edges of v must have odd labels and its $2x$ pendant vertices can't receive either odd labels or even labels.

Otherwise, the edge weight property will be affected if the $2x$ pendant vertices are odd. Vertex weight will be affected if $2x$ pendant vertices receive equal input. As a result, we lack such parity weighted TAT labeling assignment results in a discrepancy. The evidence is now complete. If $\lambda(u)$ and $\lambda(v)$ are odd, then just the odd label should be applied to $\lambda(e_1)$. Due to the odd nature of $\lambda(u)$ and $\lambda(e_1)$, we are required to have $2y$

odd incident edges for some $y \in Z^+$. Additionally, their incident pendant vertices get $2y$ odd labels. The remaining $2(m-y)$ incident edges now acquire even labels, and the $2(m-y)$ pendant vertices now receive even labels. Given that $\lambda(v)$ and $\lambda(e_1)$ are odd, we must also have $2d$ odd incident edges for any $d \in Z^+$. Additionally, the pendant vertex's second number receives an odd label exclusively. The remaining $2(m-d)$ incident edges of v are now even, and its pendant vertices $(2m-d)$ only receive an even label.

The total number of odd labels that must be assigned is $4(y+d)+3$, and the total number of even labels that must be assigned is $4(2m-y-d)$. However, the total number of possible odd labels in the range set S is equal to $4m+2$, and the total number of available even labels is equal to $4m+1$. We encounter a contradiction since the total number of necessary odd labels does not match the total number of available odd labels. If $\lambda(u)$ and $\lambda(v)$ are even, then $\lambda(e_1)$ must be odd only. Since $\lambda(u)$ and $\lambda(e_1)$ are in different parity, we have $(2z+1)$ number of incident edges of u are odd, for some $z \in Z^+$. Also, their incident $(2z+1)$ pendant vertices labels can't be assigned by either odd or even. If so, the vertex weight (or) edge weight property fails.

From all the above cases, we conclude that the bistar graph $B_{n,n}$ is not a parity weighted TAT graph for any $n \geq 1$.

□

Theorem 2.7. *The ladder graph L_n is a strong parity weighted TAT graph only when $n = 2, 3, 5, 6, 7, 8, 10$ and 39.*

Proof. Consider the ladder graph L_n for $n \geq 2$. Let $V(L_n) = \{u_i, v_i : i = 1, 2, 3, \ldots, n\}$ and $E(L_n) = \{u_iu_{i+1} : i = 1, 2, 3, \ldots, n-1\} \cup \{v_iv_{i+1} : i = 1, 2, 3, \ldots, n-1\} \cup \{u_iv_i | i = 1, 2, 3, \ldots, n\}$. Now, $p = |V(L_n)| = 2n$, $q = |E(L_n)| = 3n-2$ and $p+q = 5n-2$. Let o represent the odd label and e represent the even label.

Let $S = \{1, 2, 3, \ldots, p+q\} = \{1, 2, 3, \ldots, 5n-2\}$. Actual number of odd labels available in the set S is,

$$A_0 = \left\lceil \frac{5n-2}{2} \right\rceil = \begin{cases} \dfrac{5n-1}{2}, & \text{if} \quad n \quad \text{is odd.} \\ \dfrac{5n-2}{2}, & \text{if} \quad n \quad \text{is even.} \end{cases}$$

Actual number of even labels available in the set S is,

$$A_e = \left\lfloor \frac{5n-2}{2} \right\rfloor = \begin{cases} \dfrac{5n-3}{2}, & \text{if} \quad n \quad \text{is odd.} \\ \dfrac{5n-2}{2}, & \text{if} \quad n \quad \text{is even.} \end{cases}$$

We note that, $A_0 = A_e = \dfrac{5n-2}{2}$ if n is even, but $A_0 > A_e$ if n is odd.

Consider the following possible cases with a total labeling $\lambda : V(L_n) \cup E(L_n) \to S$,

Case (i): Suppose we consider the scheme $[\lambda(u_1), \lambda(u_1u_2), \lambda(u_2)] = [e, o, e]$. Then, $\lambda(u_1v_1) = o$, $\lambda(v_1) = e$. $[e, o, e]$ patterns repeats in $(2n-1)$ entries both in top and bottom of L_n. In this way, we start $[\lambda(u_1), \lambda(u_1u_2), \lambda(u_2)] = [e, o, e]$ but $[\lambda(u_{n-1}), \lambda(u_{n-1}u_n), \lambda(u_n)] = [e, o, e]$ in the top of L_n.

Similarly, in the bottom of the L_n, we start $[e, o, e]$ for $[\lambda(v_1), \lambda(v_1v_2), \lambda(v_2)]$ and ends with $[e, o, e]$ for $[\lambda(v_{n-1}), \lambda(v_{n-1}v_n), \lambda(v_n)]$. On the other hand, all $\lambda(u_iv_i)$ must be an odd label for all $i = 1, 2, 3, \dots, n$. Moreover, we have $(2n - 1)$ entries (vertices and edges) on the top of L_n, and also $(2n-1)$ entries on the bottom of L_n. But, in the middle, we have n entries. Now, the required number of odd labels present in the scheme $[e, o, e]$ is given by,

$$R_0 = \frac{1}{3}(2n-1) + n + \frac{1}{3}(2n-1) = \frac{1}{3}[7n-2].$$

The required number of even labels present in the scheme $[e, o, e]$ is given by,

$$R_e = \frac{2}{3}(2n-1) + \frac{2}{3}(2n-1) = \frac{1}{3}(8n-4).$$

This implies,

$$R_o + R_e = 5n - 2,$$
$$R_o + R_e = p + q = A_o + A_e.$$

Now, if n is odd, then λ is parity weighted TAT labeling only when $A_o = R_o$ and $A_e = R_e$. By solving the conditions $A_o = R_o$ and $A_e = R_e$, we get $n = -1$. Since n is not a positive integer, we do not need to consider this case. If n is even, then the conditions $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = 2$. Hence, $[e, o, e]$ scheme is valid only for L_2.

Case (ii): Suppose we consider the scheme starts with $[\lambda(u_1), \lambda(u_1u_2), \lambda(u_2)] = [e, o, e]$ and it repeats in $(2n - 3)$ entries both in top and bottom of L_n and ends with $[e, o]$ for $[\lambda(u_{n-1}u_n), \lambda(u_n)]$.

Similarly, at the bottom starts with $[e, o, e]$ for $[\lambda(v_1), \lambda(v_1v_2), \lambda(v_2)]$ and ends with $[e, o]$ for $[\lambda(v_{n-1}v_n), \lambda(v_n)]$. But, all the labels in the middle are odd. Now,

$$R_0 = \frac{1}{3}(2n-3) + 2 + \frac{1}{3}(2n-3) + n = \frac{7n}{3}, \quad \text{and}$$
$$R_e = 2\left[\frac{2}{3}(2n-3) + 1\right] = \frac{2}{3}(4n-3).$$

If n is odd, then λ becomes parity weighted TAT labeling only when $A_o = R_o$ and $A_e = R_e$, this implies we get $n = 3$. Hence, this scheme is valid only for L_3. If n is even, then λ is parity weighted TAT labeling only when $A_o = R_o$ and $A_e = R_e$. That is, λ is parity weighted TAT labeling only when $n = 6$. This scheme is valid only for L_6. Hence this scheme is valid only for L_6.

Case (iii): If we start a scheme with $[e, o, e]$ and it repeats in $(2n - 2)$ entries both at the top and bottom of L_n from left to right. The scheme ends with e only for v_n and u_n as well. All the middle edges receive o only. Now,

$$R_o = \frac{2}{3}(2n-2) + n = \frac{1}{3}[7n-4], \quad \text{and}$$
$$R_e = 2\left[\frac{2}{3}(2n-2) + 1\right] = \frac{2}{3}(4n-1).$$

If n is odd, then the conditions $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = -5$. Hence, this case of scheme is not possible for any L_n if n is odd. If n is even, then the conditions $A_o = R_o$ and $A_e = R_e$ are valid only when $n = -2$. Since n is a negative integer, this scheme is not valid for any L_n, if n is even.

Case (iv): If the pattern starts with $[o, e, e]$ and it repeats in $(2n-2)$ entries in both top and bottom of L_n, and the pattern ends with only one o in both top and bottom. All the remaining middle edges receive o only at n times. Now,

$$R_o = \frac{2}{3}(2n-2) + 2 + n = \frac{1}{3}[7n+2], \quad \text{and}$$
$$R_e = 2\left(\frac{2}{3}(2n-2)\right) = \frac{8}{3}[n-1].$$

If n is odd, then the conditions for λ to be a strong parity weighted TAT labeling are given by $A_o = R_o$ and $A_e = R_e$. By solving the above two conditions, we get $n = 7$.

Hence, this scheme is valid only for L_7. If n is even, then the conditions $A_o = R_o$ and $A_e = R_e$ for λ to be a strong weighted TAT labeling are valid only when $n = 10$. Hence, this scheme is valid only for L_{10}.

Case (v): If the pattern starts with $[o, e, e]$ and it repeats in $(2n-3)$ entries both at the top and bottom of L_n respectively. It ends with $[o, e]$ for $[u_{n-1}u_n, u_n]$ and $[v_{n-1}v_n, v_n]$. All other middle n edges receive o only. Now,

$$R_0 = 2\left(\frac{1}{3}(2n-3) + 1\right) + n = \frac{7n}{3}, \quad \text{and}$$
$$R_e = 2\left(\frac{2}{3}(2n-3) + 1\right) = \frac{2}{3}(4n-3).$$

If n is odd, then the conditions of strong parity weighted TAT labeling $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = 3$. Hence, this scheme is valid only for L_3. If n is even, then the conditions $A_o = R_o$ and $A_e = R_e$ are valid only when n=6. Hence, this scheme is valid only for L_6.

Case (vi): If the pattern starts with $[o, e, e]$ and it repeats in $(2n-1)$ entries both at the top and in the bottom of L_n respectively from left to right. All the remaining middle $n-$ edges receive o only. Now,

$$R_0 = 2\left(\frac{1}{3}(2n-1) + 1\right) + n = \frac{1}{3}(7n-2), \quad \text{and}$$
$$R_e = 2\left(\frac{2}{3}(2n-1)\right) = \frac{4}{3}(2n-1).$$

If n is odd, then the required conditions for λ to be a strong parity weighted TAT labeling are $A_o = R_o$ and $A_e = R_e$. After solving the above two conditions, we get $n = -1$. As n is a negative integer, this scheme is not possible for L_n if n is odd. If n is even, then the condition $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = 2$. Hence, this scheme is valid only for L_2.

Case (vii): If the scheme starts with $[e, e]$ on the top and $[o, o]$ on the bottom but in the middle u_1v_1 receives e all others are o. The remaining entries at the top and bottom respectively receive the pattern $[o, e, e]$ and it repeats in $(2n - 4)$ entries, the scheme ends with o only on both top and bottom respectively. Now,

$$R_0 = 2 + 2\left(\frac{1}{3}(2n-4)+1\right) + (n-1) = \frac{1}{3}(7n+1), \quad \text{and}$$
$$R_e = 2 + 2\left(\frac{2}{3}(2n-4)\right) + 1 = \frac{1}{3}(8n-7).$$

If n is odd, then the conditions $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = 5$. Hence, this scheme is valid only for L_5. If n is even, then the conditions $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = 8$ Hence, this scheme is valid only for L_8.

Case (viii): If the pattern starts with $[o, o, o, o, o, e, e, e, o, e, e]$ and it repeats for $(2n - 1)$ entries on the top and the reversal of the above pattern $[e, e, o, e, e, e, o, o, o, o, o]$ starts and repeats in the bottom for $(2n-1)$ entries. In the middle edge, we have the pattern $[e, o, e, e, o, e]$ and it repeats for n entries. Now,

$$R_0 = 2\left(\frac{6}{11}(2n-1)\right) + \frac{2}{6}n = \frac{1}{33}(83n-36), \quad \text{and}$$
$$R_e = 2\left(\frac{5}{11}(2n-1)\right) + \frac{4}{6}n = \frac{1}{33}(82n-30).$$

If n is odd, then by setting $A_o = R_o$ and $A_e = R_e$ we get $n = 39$. Hence, this scheme is valid only for L_{39}. If n is even, then the conditions $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = 6$. Hence, this scheme is valid for L_6.

Case (ix): If the scheme starts with $[o, o, o, o, o, e, e, e, o, e, e]$ at the top and it repeats for $(2n-10)$ entries and ends with $[o, o, o, o, o, e, e, e, o]$ on the top. Similarly, at the bottom, the scheme starts with $[e, e, o, e, e, e, o, o, o, o, o]$ and it repeats for $(2n-10)$ entries at the bottom and the scheme ends with $[e, e, o, e, e, e, o, o, o]$ at the bottom. In the middle, the scheme starts with $[e, o, e]$ and it repeats in $(n - 2)$ entries and ends with the scheme $[e, o]$. Now,

$$R_0 = 2\left(\frac{6}{11}(2n-10)\right) + 6 + 4 + \frac{2}{6}(n-2) + 1 = \frac{1}{33}(83n-19), \quad \text{and}$$
$$R_e = 2\left(\frac{5}{11}(2n-10)\right) + 3 + 5 + \frac{2}{3}(n-2) + 1 = \frac{1}{33}(82n-47).$$

If n is odd, then the condition $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = 5$. Hence, this scheme is valid only for L_5. If n is even, then $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = -28$. Since n is a negative integer, this scheme is not possible for any L_n when n is even.

Case (x): If the scheme starts with $[o, o, o, o, o, e, e, e, o, e, e]$ at the top and it repeats for $(2n - 6)$ entries and ends with $[o, o, o, o, o]$ at the top. In the bottom, the scheme starts with $[e, e, o, e, e, e, o, o, o, o, o]$ and it repeats for $(2n-6)$ entries and ends with $[e, e, o, e, e]$. In the middle, the scheme starts with $[e, o, e]$ and it repeats in n entries. Now,

$$R_0 = 2\left(\frac{6}{11}(2n-6)\right) + 5 + 1 + \frac{n}{3} = \frac{1}{33}(83n-18), \quad \text{and}$$
$$R_e = 2\left(\frac{5}{11}(2n-6)\right) + 4 + \frac{2}{3}(n) = \frac{1}{33}(83n-48).$$

If n is odd, then $A_o = R_o$ and $A_e = R_e$ are valid only when $n = 3$. Hence, this scheme is valid only for L_3. If n is even, then the conditions $A_o = R_o$ and $A_e = R_e$ are satisfied only when $n = -30$. Since n is a negative integer, this scheme does not exist for any L_n if n is even.

Case (xi): If we choose a different scheme than the one we previously described, then the initial fact that $A_o = R_o$ or $A_e = R_e$ is contradicted. In certain additional situations, the definitions of even vertex weight and odd edge weight prevented us from assigning o and e.

The only instances where L_n is a strong parity weighted TAT graph are those where $n = 2, 3, 5, 6, 7, 8, 10$ and 39.

□

3 Conclusions

We present a new type of labeling known as SPAT labeling, which offers a unique perspective on distinct parity conditions in graph labelings. Here we prove that cycle, path, star, bistar, complete, and ladder graphs admit such labeling. Our methodology is based on new combinatorial strategies that enforce the parity conditions while making sure all the vertex and edge weights are pairwise distinct and totally antimagic.

Strong parity-weighted labeling is one of the main consequences of these results. These labelings prove to be highly valuable, particularly in understanding the intricate relationships between graph structures and labeling constraints. Apart from the value of applying parity-weighted techniques, this paper contributes to a growing theory on antimagic labelings, which shows promise in solving more complex labeling problems of graph types.

Adding SPAT labeling to complex graph structures like bipartite graphs, hypercubes, and Cartesian products of graphs is an exciting area for future research. Another area of research involves examining the algorithmic complexity involved in determining whether any given graph is suitable for such labeling. Such a study could delve deeper into the structural properties of labeled graphs, providing a deeper understanding of their structures. It could also explore further relationships between parity-weighted labelings and graph-theoretic invariants like connectivity or chromatic number.

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